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Global dynamics of asymptotically locally AdS spacetimes with negative mass

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Abstract

The Einstein vacuum equations in 5D with negative cosmological constant are studied in biaxial Bianchi IX symmetry. We show that if initial data of Eguchi–Hanson type, modelled after the 4D Riemannian Eguchi–Hanson space, have negative mass, the future maximal development does not contain horizons, i.e. the complement of the causal past of null infinity is empty. In particular, perturbations of Eguchi–Hanson–AdS spacetimes within the biaxial Bianchi IX symmetry class cannot form horizons, suggesting that such spacetimes are potential candidates for a naked singularity to form. The proof relies on an extension principle proven for this system and *a priori* estimates following from the monotonicity of the Hawking mass.

Keywords: general relativity, cosmological constant, five dimensions, negative mass, event horizon, naked singularity, Eguchi–Hanson–AdS spacetimes

(Some figures may appear in colour only in the online journal)

1. Introduction

1.1. The Einstein vacuum equations with negative cosmological constant

The Einstein vacuum equations in n dimensions ($n > 2$)

$$\mathrm{Ric}(g) = \frac{2}{n-2}\Lambda g \tag{1.1}$$

with cosmological constant $\Lambda \in \mathbb{R}$ can be understood as a system of second-order partial differential equations for the metric tensor g of an n-dimensional spacetime $(\mathcal{M}, g)$. Solutions with negative cosmological constant $\Lambda = -(n-1)(n-2)/(2\ell^2) < 0$ have drawn considerable attention in recent years, mainly due to the conjectured instability of these spacetimes. For more details, see [And06, DH06a, BR11, DHS11, DHMS12, HLSW15] and references therein.

The system (1.1) is of hyperbolic nature, and studying the dynamic evolution of initial data is very difficult in general, leading us to take recourse to settings with high degrees of symmetry. In particular, it is desirable to reduce the dimension of the dynamical problem to the simplest case of $1+1$ dimensions. This approach has a longer history for $\Lambda = 0$. There, for $n = 4$, the only symmetry group achieving the reduction to a $1+1$-dimensional problem whilst consistent with the spacetime being asymptotically flat is spherical symmetry. However, the well-known Birkhoff theorem prevents any dynamical consideration since such a 4D spacetime necessarily embeds locally into a subset of a member of the Schwarzschild family, and this embedding is isometric.

To study spherically symmetric gravitational dynamics in 4D, one can follow the approach of the seminal work by Christodoulou and couple gravity to matter. In a sequence of papers—see his own survey article [Chr99] for references—he initiated the rigorous analysis of spherically symmetric gravitational collapse for $\Lambda = 0$ by studying the Einstein-scalar field system. The model of a real massless scalar field was chosen because, on the one hand, this matter model does not develop singularities in the absence of gravity and, on the other hand, its wave-like character resembles the character of general gravitational perturbations of Minkowski space. Christodoulou's work led to a complete understanding of weak and strong cosmic censorship for this model. His approach has later been extended to other matter models; see [Kom13] for a systematic overview and references.

Christodoulou's approach was adapted to the context of $\Lambda < 0$ by Holzegel and Smulevici in [HS11a] and by Holzegel and Warnick in [HW13], who show well-posedness of the Einstein–Klein–Gordon system with the scalar field satisfying various reflecting boundary conditions at infinity. The work [HS11b] shows stability of Schwarzschild–AdS in this symmetry class for Dirichlet boundary conditions. A recent breakthrough has been achieved by Moschidis in [Mos17a] and [Mos17b]; in his work, he shows instability of exact anti-de Sitter space as a solution to the Einstein-null dust system in spherical symmetry with an inner mirror.

Another possibility of evading the restrictions of Birkhoff's theorem is to study (1.1) in higher dimensions. Working in 5D and imposing biaxial Bianchi IX symmetry, a symmetry corresponding to a subgroup of $SO(4)$, still reduces the system to $1+1$ dimensions and introduces a dynamical variable B, not dissimilar to the scalar field in the coupled system. This model was introduced by Bizón, Chmaj and Schmidt. In [BCS05], they initiated the study of gravitational collapse for $\Lambda = 0$ in this symmetry class by numerical computations; investigations along those lines were continued in [DH06b] and [BCS06].

The study of this system in the realm of negative Λ has been initiated by Dafermos and Holzegel in 2006. In [DH06a]—now mostly cited for the conjecture of the instability of exact AdS space—, our corollary 1.11 has been put forward without rigorous proof. Back then, the problem of proving local well-posedness for the system in biaxial Bianchi IX symmetry was not solved, thus no extension principle sufficiently strong was available. The present paper can be seen as a completion of [DH06a], building on the insight into problems in asymptotically locally AdS spacetimes obtained over the past decade; for an overview of this work, see [HW13, EK14] and references therein.

In 5D and for $\Lambda < 0$, (1.1) has many static solutions which are asymptotically locally AdS. A spacetime is asymptotically locally AdS if the asymptotics of the metric towards conformal infinity $\mathcal{I}$ is modelled after AdS space, but $\mathcal{I}$ need not be $\mathbb{R} \times S^3$ topologically. Prominent examples of such static solutions are exact AdS_5 space with spherical conformal infinity[1] and the AdS soliton of [HM98] with toric $\mathcal{I}$. Eguchi–Hanson–AdS spacetimes form another such family with $\mathcal{I} \cong \mathbb{R} \times (S^3/\mathbb{Z}_n)$ for $n \geqslant 3$[2]. Eguchi–Hanson–AdS spacetimes have negative mass at infinity (proposition 1.9) and additionally, they are conjectured to have minimal mass among all asymptotically locally AdS spacetimes with topology $\mathcal{I} \cong \mathbb{R} \times (S^3/\mathbb{Z}_n)$ (conjecture 1.13). In other words, Eguchi–Hanson–AdS spacetimes are conjectured to be the energy 'ground state' solutions given this topology at infinity. This is discussed in more detail in section 1.4.

1.2. *Spaces of Eguchi–Hanson type*

Riemannian Eguchi–Hanson space is isometrically isomorphic to the four dimensional manifold $(a, \infty) \times (S^3/\mathbb{Z}_2)$—for any fixed $a > 0$—equipped with the metric

$$g = \frac{1}{1 - \frac{a^4}{\varrho^4}} \mathrm{d}\varrho^2 + \frac{\varrho^2}{4} \left(\sigma_1^2 + \sigma_2^2\right)^2 + \frac{\varrho^2}{4} \left(1 - \frac{a^4}{\varrho^4}\right) \sigma_3^2.$$

The left-invariant one-forms σ_i are discussed below. The apparent singularity at $\varrho = a$ is removable. Eguchi–Hanson space exhibits an $SU(2) \times U(1)$ symmetry and can be considered as being defined on the cotangent bundle of the two-sphere.

In the present context, we introduce 4D Riemannian manifolds modifying Eguchi–Hanson space to the asymptotically locally AdS context. These will serve as initial data for the 5D Einstein vacuum equations

$$\mathrm{Ric}(g) = \frac{2}{3} \Lambda g \tag{1.2}$$

via the local well-posedness theorem 1.7. Our data also exhibit an $SU(2) \times U(1)$ symmetry, thus giving rise to spacetimes with biaxial Bianchi IX symmetry. Then Eguchi–Hanson–AdS spacetimes form particular examples of the spacetimes thus obtained.

Definition 1.1. We say that an initial data set $(\mathcal{S}, \overline{g}, K)$ to (1.2) exhibiting $SU(2) \times U(1)$ symmetry is of Eguchi–Hanson type if

$$\mathcal{S} = (a, \infty) \times \left(S^3/\mathbb{Z}_n\right)$$

for fixed $a > \ell$ and $n \geqslant 3$ satisfying

$$\frac{n^2}{4} = 1 + \frac{a^2}{\ell^2},$$

and if

$$\overline{g} = \frac{1}{A} \mathrm{d}\varrho^2 + \gamma \quad \text{with} \quad \gamma = \frac{1}{4} r^2 \mathrm{e}^{2B} \left(\sigma_1^2 + \sigma_2^2\right) + \frac{1}{4} r^2 \mathrm{e}^{-4B} \sigma_3^2,$$

[1] Numerical studies within the biaxial Bianchi IX symmetry class for perturbations of AdS_5 were carried out recently in [BR17].

[2] The space $S^3/\mathbb{Z}_n$ is defined in the usual way as the lense space $L(n, 1)$.

where $(\sigma_1, \sigma_2, \sigma_3)$ is a basis of left-invariant one-forms on $SU(2)$ (see below) and

$$A,\, r,\, B:\ (a, \infty) \to \mathbb{R}$$

are smooth functions such that the following conditions hold:

(i) Around the centre $\varrho = a$, the functions satisfy the regularity conditions

$$\begin{aligned} A =& \frac{n^2}{a}(\varrho - a) + \mathcal{O}\left((\varrho - a)^2\right) \\ r =& 2^{1/3} a^{5/6} (\varrho - a)^{1/6} + \mathcal{O}\left((\varrho - a)^{7/6}\right) \\ B + \log r =& \log a + \mathcal{O}((\varrho - a)). \end{aligned}$$

(ii) The function A is non-zero on (a, ∞). Moreover

$$A = \frac{\varrho^2}{\ell^2} + 1 + o(1)$$

as $\varrho \to \infty$.

(iii) The function r is the radius of the topological 3-spheres at ϱ. Moreover

$$r = \varrho + \mathcal{O}(1)$$

as $\varrho \to \infty$.

(iv) For an $R > a$, we have

$$\int_R^\infty \left(\varrho^3 B^2 + \varrho^7 \left(\partial_\varrho B\right)^2\right) \mathrm{d}\varrho < C \ \text{ and } \sup_{\varrho \in (R, \infty)} |\varrho^3 \partial_\varrho B| < C.$$

We require that

$$M := \lim_{\varrho \to \infty} \left(\frac{r^2}{2} \left[1 + \frac{r^2}{12}\left((\mathrm{tr}_\gamma K)^2 - H^2\right)\right] + \frac{r^4}{2\ell^2} \right),$$

where H is the mean curvature of the symmetry orbits, is finite; we call M the mass of $(\mathcal{S}, \overline{g}, K)$ at infinity. Recall that in the expression $\mathrm{tr}_\gamma K$, K is restricted to act only on vectors tangent to the topological three-spheres.

Remark 1.2.

1. The left-invariant one-forms satisfy

$$\mathrm{d}\sigma_1 + \sigma_2 \wedge \sigma_3 = 0,\ \mathrm{d}\sigma_2 + \sigma_3 \wedge \sigma_1 = 0,\ \mathrm{d}\sigma_3 + \sigma_1 \wedge \sigma_2 = 0.$$

One can choose Euler angles $(\vartheta, \varphi, \psi)$, $0 < \vartheta < \pi$, $0 \leqslant \varphi < 2\pi$, $0 \leqslant \psi < 4\pi$ on $SU(2)$ such that

$$\sigma_1 = \sin\vartheta\, \sin\psi\, \mathrm{d}\varphi + \cos\psi\, \mathrm{d}\vartheta,\ \sigma_2 = \sin\vartheta\, \cos\psi\, \mathrm{d}\varphi - \sin\psi\, \mathrm{d}\vartheta,\ \sigma_3 = \cos\vartheta\, \mathrm{d}\varphi + \mathrm{d}\psi.$$

In terms of the left-invariant one-forms, the Minkowski metric on $\mathbb{R}^5$ is given by

$$g_{\mathrm{Mink}} = -\mathrm{d}t^2 + \mathrm{d}r^2 + \frac{1}{4} r^2 \left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right).$$

The Euler angles $(\vartheta, \varphi, \psi)$ parametrise the 3-sphere away from the poles. By restricting ψ to have period $4\pi/n$, we obtain coordinates on $S^3/\mathbb{Z}_n$.

2. By identity (A.1), the notion of mass at infinity is consistent with the renormalised Hawking mass introduced in definition 1.5.
3. The triple $(\mathcal{S}, \overline{g}, K)$ is asymptotically locally AdS, consistent with definition 1.6.

Prima facie it seems as if $\overline{g}$ had a singularity at $\varrho = a$. However, one should compare this situation to that of spherical symmetry in spherical coordinates. This intuition is made more precise in the following

Proposition 1.3. *Let $(\mathcal{S}, \overline{g}, K)$ be of Eguchi–Hanson type. Then there is a $b > 0$ such that for $\varrho \in (a, b)$, $(\mathcal{S} \cap \{\varrho < b\}, \overline{g}|_{\{\varrho<b\}})$ has topology $\mathbb{R}^2 \times S^2$ and can be smoothly extended by adding a 2-sphere at $\varrho = a$. The resulting manifold is smooth and has no boundary.*

Proof. Define

$$z := \frac{4\sqrt{a}}{n}\,(\varrho - a)^{1/2}\,.$$

To leading order, we have

$$r^2 \mathrm{e}^{-4B} \sim \frac{r^6}{a^4} \sim 4a\,(\varrho - a) = \frac{1}{4} n^2 z^2$$

around $\varrho = a$. Thus the metric becomes

$$\overline{g} \sim \frac{1}{4}\left(\mathrm{d}z^2 + n^2 z^2\,(\mathrm{d}\psi + \cos\vartheta \mathrm{d}\varphi)^2\right) + \frac{\varrho^2}{4}\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2\right)$$

to leading order. For fixed (ϑ, φ), the restriction on the range of ψ (see remark 1.2) guarantees that the metric can be extended smoothly to $\varrho = a$. By adding an S^2 at $\varrho = a$, we obtain a manifold without boundary that has local topology $\mathbb{R}^2 \times S^2$. □

Remark 1.4. We immediately see that at infinity, $(\mathcal{S}, \overline{g})$ is asymptotically locally AdS.

Given Eguchi–Hanson-type initial data, the Einstein vacuum equation (1.2) are well-posed in the biaxial Bianchi IX symmetry class—see [Dol17] for a proof. We merely state the well-posedness theorem here.

Definition 1.5. Let $(\mathcal{M}, g)$ be a 5D spacetime. Then $(\mathcal{M}, g)$ exhibits a biaxial Bianchi IX symmetry if, topologically,

$$\mathcal{M} = \mathcal{Q} \times \left(S^3/\Gamma\right),$$

for $\mathcal{Q}$ a 2D manifold (possibly with boundary) and Γ a discrete group with $\Gamma \in \{\emptyset, \mathbb{Z}_2, \mathbb{Z}_3, \ldots\}$, such that

$$g = h + \frac{1}{4} r^2 \left(\mathrm{e}^{2B}(\sigma_1^2 + \sigma_2^2) + \mathrm{e}^{-4B}\sigma_3^2\right). \tag{1.3}$$

Here h is a Lorentzian metric on $\mathcal{Q}$, and r and B are smooth real-valued functions on $\mathcal{Q}$. The value $r(q)$ is the area radius of the squashed sphere through $q \in \mathcal{Q}$, i.e.

$$2\pi^2 r^3 = \mathrm{vol}\left(S_q^3\right),$$

where S_q^3 is the sphere at q. In this symmetry class, we introduce the renormalised Hawking mass (henceforth referred to as Hawking mass)

$$m: \mathcal{Q} \to \mathbb{R}$$

by

$$m = \frac{r^2}{2}\left(1 - g\left(\nabla r, \nabla r\right)\right) + \frac{r^4}{2\ell^2}.$$

Definition 1.6. A spacetime $(\mathcal{M}, g)$ exhibiting biaxial Bianchi IX symmetry is asymptotically locally AdS with radius ℓ and conformal infinity $\mathcal{I}$ if it is conformally equivalent to a manifold $(\tilde{\mathcal{M}}, \tilde{g})$ with boundary $\mathcal{I} := \partial\tilde{\mathcal{M}}$ such that

(i) Conformal infinity $\mathcal{I}$ has topology $\mathbb{R} \times (S^3/\Gamma)$.
(ii) The inverse $\tilde{r} := r^{-1}$ is a boundary defining function for $\mathcal{I}$, i.e. $\tilde{r} = 0$ and $\mathrm{d}\tilde{r} \neq 0$ on $\mathcal{I}$.
(iii) The rescaled metric $r^{-2}g$ is a smooth metric on a neighbourhood of $\mathcal{I}$ in $\tilde{\mathcal{M}}$.
(iv) For small $\tilde{r} > 0$, there exist coordinates $(t, \tilde{r})$ on $\mathcal{Q}$ such that, locally,

$$h\left(\frac{1}{\tilde{r}^2}\partial_{\tilde{r}}, \frac{1}{\tilde{r}^2}\partial_{\tilde{r}}\right) = \ell^2\tilde{r}^2 + \mathcal{O}\left(\tilde{r}^4\right).$$

in a neighbourhood of $\mathcal{I}$.
(v) The quantity B satisfies a Dirichlet boundary condition, i.e. $B = 0$ on $\mathcal{I}$.

Theorem 1.7. *Let $(\mathcal{S}, \overline{g}, K)$ be an initial data set with mass M at infinity such that $(\mathcal{S}, \overline{g}, K)$ is of Eguchi–Hanson type. Then there is a $T > 0$ and a manifold $\mathcal{M} := (-T, T) \times \mathcal{S}$ equipped with a metric g exhibiting biaxial Bianchi IX symmetry such that $(\mathcal{M}, g)$ is asymptoticall locally AdS, g solves (1.2) and $\{0\} \times \mathcal{S}$ has induced metric $\overline{g}$ and second fundamental form K. Moreover, $(\mathcal{M}, g)$ is the unique asymptotically locally AdS solution to (1.2) with initial data $(\mathcal{S}, \overline{g}, K)$.*

Remark 1.8.

1. The local well-posedness theorem for an initial data set $(\mathcal{S}, \overline{g}, K)$ yields the existence of a unique maximal development in the sense of [HS11a], see figure 1.
2. The proof of the local well-posedness theorem in [Dol17] proceeds along the lines of [HW13]. Well-posedness of the Einstein vacuum equations for $\Lambda < 0$ in 4D without symmetry assumptions was shown by Friedrich in [Fri95], and a recent generalisation to higher dimensions by Enciso and Kamran is also available; see [EK14]. In particular, theorem 1.7 follows from their work. However the theorem as stated is too general for an extension principle (section 3.2); so to exploit the monotonicity of the Hawking mass (see section 3.3), a local well-posedness result in norms propagated by the mass (as in section 2) is required. It was therefore more convenient to reprove the local well-posedness result in the biaxial Bianchi IX symmetry class.
3. As explained in [Dol17], the most difficult part of this theorem is to establish a local well-posedness result around null infinity with initial data on an ingoing null hypersurface. This local result is reviewed in section 2.

The explicit examples behind this well-posedness theorem are Eguchi–Hanson–AdS spacetimes, constructed in [CM06]. They form a family of solutions $(\mathcal{M}_{\mathrm{EH},a}, g_{\mathrm{EH},a})$ to (1.1) in 5D, where $a > \ell$ satisfies

$$1 + \frac{a^2}{\ell^2} = \frac{n^4}{2}$$

for a natural number $n \geqslant 3$. For fixed $\Lambda = -6/\ell^2 < 0$, they form a one-parameter family of static spacetimes exhibiting biaxial Bianchi IX symmetry. If we define

$$r = \varrho \left(1 - \frac{a^4}{\varrho^4}\right)^{1/6}, \quad \Omega^2 = 1 + \frac{\varrho^2}{\ell^2}, \quad B = -\frac{1}{6} \log\left(1 - \frac{a^4}{\varrho^4}\right)$$

and choose coordinates such that

$$h = -\frac{1}{2}\Omega^2 \left(\mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}u\right),$$

with

$$\mathrm{d}u = \mathrm{d}t - \frac{1}{\left(1 + \frac{\varrho^2}{\ell^2}\right)\left(1 - \frac{a^4}{\varrho^4}\right)^{1/2}}\mathrm{d}\varrho, \quad \mathrm{d}v = \mathrm{d}t + \frac{1}{\left(1 + \frac{\varrho^2}{\ell^2}\right)\left(1 - \frac{a^4}{\varrho^4}\right)^{1/2}}\mathrm{d}\varrho,$$

the metric takes the form

$$\begin{aligned} g_{\mathrm{EH},a} = &-\left(1 + \frac{\varrho^2}{\ell^2}\right)\mathrm{d}t^2 + \frac{1}{\left(1 + \frac{\varrho^2}{\ell^2}\right)\left(1 - \frac{a^4}{\varrho^4}\right)}\mathrm{d}\varrho^2 \\ &+ \frac{\varrho^2}{4}\left(1 - \frac{a^4}{\varrho^4}\right)\left(\mathrm{d}\psi + \cos\vartheta\,\mathrm{d}\varphi\right)^2 + \frac{\varrho^2}{4}\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right) \end{aligned}$$

in $(t, \varrho, \vartheta, \varphi, \psi)$ variables with $\varrho \in (a, \infty)$. In the limit $\ell \to \infty$, the metric $g_{\mathrm{EH},a}$ restricted to hypersurfaces of constant t yields the Riemannian Eguchi–Hanson metric, which was first presented in [EH79].

We immediately note:

Proposition 1.9. *Let $(\mathcal{M}_{\mathrm{EH},a}, g_{\mathrm{EH},a})$ be an Eguchi–Hanson–AdS spacetime. Then*

$$M_{\mathrm{EH},a} := \lim_{\varrho\to\infty} m = -\frac{5}{6}\frac{a^4}{\ell^2}$$

is negative. At the centre $\varrho = a$, the Hawking mass is ill-defined, tending to $-\infty$.

1.3. The main result

The main novel result of this paper consists in showing that for initial data of Eguchi–Hanson type with negative mass, a Penrose diagram such as figure 2 cannot arise.

Theorem 1.10. *Let $(\mathcal{S}, \overline{g}, K)$ be of Eguchi–Hanson type with negative mass $M < 0$ at infinity. Then there is no future horizon in the maximal development, i. e. the complement of the causal past of null infinity is empty.*

Corollary 1.11. *Small perturbations of Eguchi–Hanson–AdS spacetimes do not contain future horizons.*

It is important to stress that the absence of a horizon is a stronger statement than the absence of trapped surfaces—shown in proposition 3.2—for a horizon concerns the causal past of null

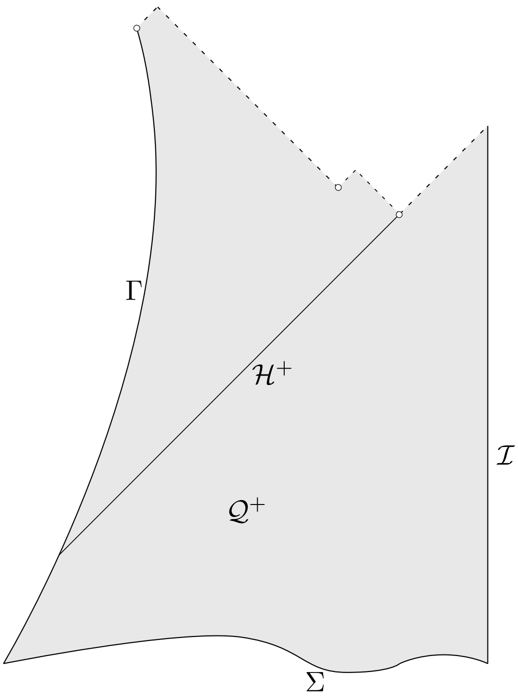

Figure 1. A general Penrose diagram of the future maximal development of initial data of Eguchi–Hanson type via theorem 1.7.

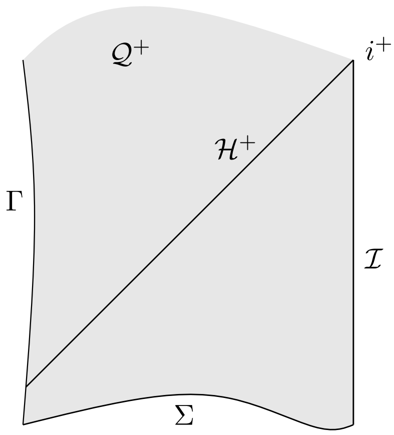

Figure 2. The Penrose diagram depicts the situation of a horizon. The endpoint of $\mathcal{I}$, denoted by i^+, is either in $\mathcal{I}$ or its completion; the latter case corresponds to future complete $\mathcal{I}$, the former one to an incomplete $\mathcal{I}$. We will show the absence of a horizon, that is the impossibility of either of those two cases.

infinity and hence the global geometry of the spacetime, whereas a trapped surface is a local phenomenon. For example, in the extended Schwarzschild solution with $\Lambda = 0$, there is a family of Cauchy surfaces coming arbitrarily close to the singularity $r = 0$ such that there exist no trapped surfaces lying in the causal past of each surface, see the classical paper by Wald and Iyer [WI91].

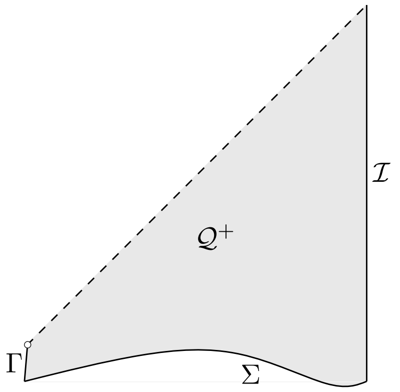

Figure 3. The Penrose diagram depicts the formation of a first singularity emanating from the central wordline Γ. The singularity is denoted by the circled dot; the dotted line is not part of the spacetime.

Combining theorem 1.10 with the arguments in section 3.1 leaves us with the following dichotomy: either the future development of Eguchi–Hanson-type data with negative mass contains a first singularity in $\overline{\Gamma}\backslash\Gamma$, where Γ is the centre—see figure 3—, or no first singularities form at all.

In virtue of the properties and conjectures described in the next section, our result, restricted to perturbations of Eguchi–Hanson–AdS spacetimes, can corroborate the conjecture put forward in in [DH06a]:

Conjecture 1.12. Small perturbations of Eguchi–Hanson–AdS spacetimes have a Penrose diagram as depicted in figure 3. Moreover, $\mathcal{I}$ is future incomplete.

In contrast, in a comparable context where no horizons can form, the work [BJ13], described in the next section, allows for growth of perturbations and global existence of the solution without the formation of a naked singularity. Thus the dynamics is very complicated and the question of the formation of naked singularities remains open.

1.4. The significance of Eguchi–Hanson–AdS spacetimes

The main motivation that sparked recent interest in asymptotically locally AdS solutions to the Einstein vacuum equations within the physics community is a putative connection between spacetimes of this form and conformal field theories defined on their respective boundaries: the AdS-CFT correspondence. It is of interest to understand what the positivity of gravitational energy means in the conformal field theory and thus 'ground states', lowest energy configurations classically allowed, deserve consideration—see [GSW02] for more details and references on the issue of gravitational energy in this context.

A ground state depends heavily on the topology at infinity. If the spacetime is asymptotically AdS, this ground state is exact anti-de Sitter space with vanishing mass—see [BGH84]. For asymptotically locally AdS spacetimes with toroidal topology at infinity, the works [HM98] and [GSW02] lend support to the conjecture that the so-called AdS soliton is the ground state in a suitable class of spacetimes.

The article [CM06] was motivated by searching for a spacetime that asymptotically approaches AdS_5/Γ, where Γ is any freely acting discrete group of isometries, but has energy less than that of AdS_5/Γ. This led to the Eguchi–Hanson–AdS solution in 5D. These spacetimes have also been conjectured in [CM06] to have minimal mass among asymptotically locally AdS spacetimes with topology $\mathrm{AdS}_5/\mathbb{Z}_n$ at infinity:

Conjecture 1.13. Let $(\mathcal{S}, \overline{g}, K)$ be of Eguchi–Hanson type with $\mathcal{S} = (a, \infty) \times (S^3/\mathbb{Z}_n)$, then

$$M \geqslant M_{\mathrm{EH},a}$$

with equality if and only if the data agree with those induced by the Eguchi–Hanson–AdS spacetime with parameter a.

In a neighbourhood of Eguchi–Hanson–AdS solutions, this was indeed shown to be true:

Theorem 1.14 ([DH06a], see also [CM06]). *Given any $a > 0$, assume initial data $(\mathcal{S}, \overline{g}, K)$ of Eguchi–Hanson type with $\mathcal{S} = (a, \infty) \times (S^3/\mathbb{Z}_n)$ which are a sufficiently small, but non-zero perturbation of the data induced by the Eguchi–Hanson–AdS spacetime with parameter a, then the mass M at infinity satisfies*

$$M_{\mathrm{EH},a} < M < 0. \tag{1.4}$$

Motivated by the static uniqueness theorem for exact AdS space [BGH84], one conjectured—see [DH06a]:

Conjecture 1.15. There are no static, globally regular asymptotically locally AdS solutions to (1.2) with topology $S^3/\mathbb{Z}_n$ with mass M satisfying (1.4).

Thus, fixing a, the Eguchi–Hanson–AdS spacetime satisfying (1.4) can be seen as the ground state in the biaxial Bianchi IX symmetry class. There is a folklore statement that such ground states would be stable under gravitational perturbations. However, in contrast, the present work, paired with the above conjectures, heuristically hints at an instability: Perturbing an Eguchi–Hanson–AdS spacetime slightly increases its mass at infinity, whilst remaining negative; therefore, by corollary 1.11, the future maximal development cannot contain a black hole, but by conjecture 1.15, there is no static end state for the perturbation, which intimates that a first singularity forms, emanating from the centre. Therefore, such perturbations are potential candidates for examples of the formation of naked singularities.

It is interesting to note that a dual situation is found for perturbations of AdS_3, as investigated in [BJ13]. There, small perturbations of 3D AdS space were studied numerically as solutions to the Einstein-scalar field system. The parallel to our case is that in 3D, there exists a mass threshold below which no black holes can form. In contrast, while the numerical computations of [BJ13] suggest turbulence which cannot be terminated by a black hole formation, they provide evidence that small perturbations remain globally regular in time since the turbulence is too weak.

Finally, studying 5D static spacetimes for various values of Λ or, more precisely, classifying their 4D Riemannian counterparts is still an active field of research in geometry. It is known that there are exactly four complete non-singular 4D Ricci flat Riemannian spaces: Euclidean space, Eguchi–Hanson space, self-dual Taub-NUT space and Taub-Bolt space. See [Gib05] for further details. Moreover, Eguchi–Hanson space has been used in geometric gluing constructions; see [Biq13] and [BK17]. For more results in this realm, both classical and recent, see [BGPP78, LeB88, EH79, BK17] and references therein.

1.5. Outline of the paper

From the local well-posedness theorem (theorem 1.7), we obtain the existence of a maximal development of Eguchi–Hanson-type data, with B satisfying a Dirichlet boundary condition at

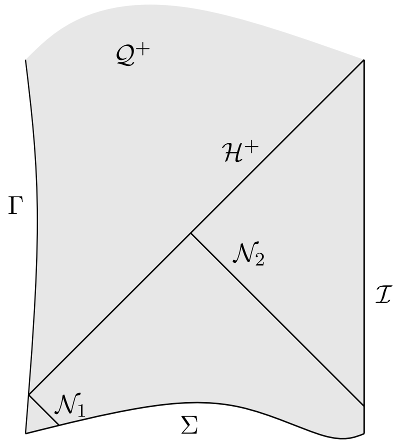

Figure 4. We can achieve that the initial data slice touches null infinity and does not reach Γ by moving from a slice such as $\mathcal{N}_1$ to $\mathcal{N}_2$.

infinity. The global geometry of spacetimes arising from such data is described in section 3.1. We also prove in that section that the spacetime is either globally regular without a horizon, or forms a horizon, or evolves into a first singularity at the centre. We proceed to show that no horizons can form in the dynamical evolution.

Proving the absence of horizons will take the structure of an argument by contradiction (section 3.3). Suppose that the Penrose diagram of the spacetime looks like figure 4. By soft arguments relying on the well-posedness result, we show that one can always find a null hypersurface such as $\mathcal{N}_2$ that does not intersect the initial hypersurface, but reaches from the horizon to null infinity. In section 3.2, an extension principle is shown for triangular regions around null infinity—such as the one enclosed by $\mathcal{H}^+$, $\mathcal{N}_2$ and $\mathcal{I}$ –, which permits to extend the solutions to the future to a strictly larger triangle, provided uniform bounds hold in the triangular region. The proof of the extension principle uses the local existence result proved in [Dol17] and reviewed in section 2. We then proceed (section 3.3) to establish that those quantities can indeed be bounded only in terms of their values on $\mathcal{N}_2$, where they hold by compactness of $\mathcal{N}_2$ and the local well-posedness result. Thus we can extend the solution along $\mathcal{I}$ beyond $\mathcal{H}^+$, which is a contradiction.

2. Local well-posedness

Before beginning with the proof of the main result, we review the local well-posedness result around null infinity of [Dol17] since the extension principle of section 3.2 relies on it. The exposition of this section parallels that of [HW13]. This will allow the reader familiar with that argument to gain quick access to the problem at hand.

Proving local well-posedness in the context of negative cosmological constant around infinity has been achieved for the 4D Einstein–Klein–Gordon system in [HS11a] and [HW13]. Several differences arise in the present context, which are outlined in [Dol17]. However, we can follow the general strategy of [HW13]. We first define the triangle

$$\Delta_{\delta,u_0} := \{(u,v) \in \mathbb{R}^2 \,:\, u_0 \leqslant v \leqslant u_0 + \delta,\, v < u \leqslant u_0 + \delta\}$$

and the conformal boundary

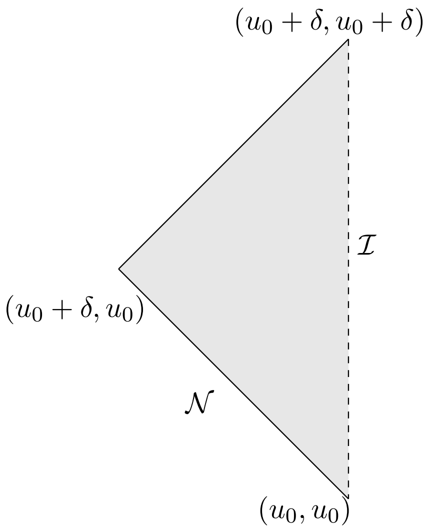

Figure 5. The triangular domain Δ_{δ,u_0} of local existence .

$$\mathcal{I} := \overline{\Delta_{\delta,u_0}} \backslash \Delta_{\delta,u_0}.$$

See figure 5 for a visualisation. Our dynamical variables are

$$(\tilde{r}, m, B) : \ \Delta_{\delta,u_0} \to \mathbb{R}_+ \times \mathbb{R}^2.$$

We treat these as defining auxiliary variables

$$r := \frac{1}{\tilde{r}}, \quad 1-\mu := 1 - \frac{2m}{r^2} + \frac{r^2}{\ell^2}, \quad \Omega^2 := -\frac{4r^4\tilde{r}_u\tilde{r}_v}{1-\mu}. \tag{2.1}$$

Here and henceforth, subscripts u and v signify partial derivatives with respect to u and v, respectively.

The general discussion of appendix A.2 show that the correct notion of a solution to the Einstein vacuum equations in the triangular region is encapsulated in the following

Definition 2.1. A weak solution to the Einstein vacuum equations in Δ_{δ,u_0} is a triple

$$(\tilde{r}, m, B) \in C^1_{\text{loc}}(\Delta_{\delta,u_0}) \cap W^{1,1}_{\text{loc}}(\Delta_{\delta,u_0}) \cap H^1_{\text{loc}}(\Delta_{\delta,u_0})$$

such that $\tilde{r}_{uu}, B_u, m_u \in C^0_{\text{loc}}$ and the equations

$$\tilde{r}_{uv} = \Omega^2\tilde{r}^3\left(-1 + \frac{1}{3}R - 2m\tilde{r}^2\right) \tag{2.2}$$

$$\partial_u m = -\frac{\tilde{r}_u}{\tilde{r}^3}\left(1 - \frac{2}{3}R\right) + \frac{4}{\Omega^2}\frac{\tilde{r}_v}{\tilde{r}^5}\left(B_u\right)^2 \tag{2.3}$$

$$\partial_v m = -\frac{\tilde{r}_v}{\tilde{r}^3}\left(1 - \frac{2}{3}R\right) + \frac{4}{\Omega^2}\frac{\tilde{r}_u}{\tilde{r}^5}\left(B_v\right)^2 \tag{2.4}$$

$$B_{uv} = \frac{3}{2}\frac{\tilde{r}_u}{\tilde{r}}B_v + \frac{3}{2}\frac{\tilde{r}_v}{\tilde{r}}B_u - \frac{1}{,}3\Omega^2\tilde{r}^2\left(\mathrm{e}^{-2B} - \mathrm{e}^{-8B}\right) \tag{2.5}$$

where

$$R = 2\mathrm{e}^{-2B} - \frac{1}{2}\mathrm{e}^{-8B},$$

are satisfied in the interior of Δ_{δ,u_0} in a weak sense.

Remark 2.2. The function spaces C^k_{loc}, $W^{k,p}_{\mathrm{loc}}$ and H^k_{loc} are the local versions of the spaces C^k, $W^{k,p}$ and H^k, respectively.

Equations (2.2), (2.4) and (2.5) are treated as the dynamical equations, whereas (2.3) can be treated as a constraint equation that is propagated.

Definition 2.3. Let $\mathcal{N} = (u_0, u_1]$. A triple

$$(\bar{\tilde{r}}, M, \overline{B}) \in C^2(\mathcal{N}) \times \mathbb{R} \times C^1(\mathcal{N})$$

is a free data set if the following hold:

(i) $\bar{\tilde{r}} > 0$ and $\bar{\tilde{r}}_u > 0$ in $\mathcal{N}$, as well as $\lim_{u\to u_0}\bar{\tilde{r}}(u) = 0$, $\lim_{u\to u_0}\bar{\tilde{r}}_u = 1/2$ and $\lim_{u\to u_0}\bar{\tilde{r}}_{uu} = 0$.
(ii) There is a constant C_0 such that

$$\int_{u_0}^{u_1}\frac{1}{(u-u_0)^3}\left[\overline{B}^2 + (\overline{B}_u)^2\right]\mathrm{d}u < C_0$$
$$\sup_{\mathcal{N}}\left|\bar{\tilde{r}}^{-2}\overline{B}\right| + \sup_{\mathcal{N}}\left|\bar{\tilde{r}}^{-1}\partial_u\overline{B}\right| < C_0.$$

From this, we obtain a complete initial data set $(\bar{\tilde{r}}, \overline{B}, \overline{m}, \bar{\tilde{r}}_v) \in C^2(\mathcal{N}) \times C^1(\mathcal{N}) \times C^1(\mathcal{N}) \times C^1(\mathcal{N})$. First, we integrate

$$\partial_u\overline{m} = -\frac{\bar{\tilde{r}}_u}{\bar{\tilde{r}}^3}\left(1 - \frac{2}{3}\overline{R}\right) - \frac{1}{\bar{\tilde{r}}\bar{\tilde{r}}_u}\left(1 - 2\overline{m}\bar{\tilde{r}}^2 + \frac{1}{\ell^2\bar{\tilde{r}}^2}\right)(\overline{B}_u)^2 \tag{2.6}$$

with boundary condition

$$\lim_{u\to u_0}\overline{m} = M$$

for the Hawking mass M at infinity. Note here that

$$1 - \frac{2}{3}\overline{R} = 8\overline{B}^2 + \mathcal{O}(\overline{B}^3),$$

whence the first term on the right hand side of (2.6) is regular as $\tilde{r} \to 0$ due to the bounds on $\overline{B}$. The function $\overline{\tilde{r}_v}$ is obtained from solving the ODE

$$\frac{\partial_u\overline{\tilde{r}_v}}{\overline{\tilde{r}_v}} = -\frac{4\bar{\tilde{r}}_u}{\bar{\tilde{r}} - 2\overline{m}\bar{\tilde{r}}^3 + \frac{1}{\ell^2\bar{\tilde{r}}}}\left(-1 + \frac{\overline{R}}{3} - 2\overline{m}\bar{\tilde{r}}^2\right)$$

with boundary condition

$$\lim_{u\to u_0}\overline{\tilde{r}_v} = -\frac{1}{2}.$$

Theorem 2.4. *Let $(\bar{\tilde{r}}, M, \overline{B})$ be a free data set in the sense of definition 2.3 on $\mathcal{N} = (u_0, u_1]$. Then there is a $\delta > 0$ such that there exists a unique weak solution $(\tilde{r}, m, B)$ of the Einstein equations in the triangle Δ_{δ,u_0} such that*

(i) $\tilde{r}$ satisfies the boundary condition

$$\tilde{r}\big|_{\mathcal{I}} = 0$$

(ii) B satisfies the boundary condition

$$B\big|_{\mathcal{I}} = 0$$

in a weak sense

(iii) $\tilde{r}$ and B agree with $\bar{\tilde{r}}$ and $\overline{B}$ when restricted to $\mathcal{N}$.

Remark 2.5. Imposing higher regularity on the initial data, we obtain a classical solution to (1.2) in Δ_{δ,u_0}; see [Dol17] for a precise statement.

We conclude this section with a remark about the Hawking mass. The mass is a dynamical variable and does not have to be conserved at infinity *a priori*. However, a geometric version of conservation holds:

Proposition 2.6. *Let $(\tilde{r}, m, B)$ be a classical solution. Set*

$$\mathcal{T} := \frac{1}{\Omega^2}\left(r_v\partial_u - r_u\partial_v\right)$$
$$\mathcal{R} := \frac{1}{\Omega^2}\left(r_v\partial_u + r_u\partial_v\right).$$

Then $\mathcal{T}$ and $\mathcal{R}$ are invariant under a change of the (u, v) coordinates that preserves the form of the metric and

$$\mathcal{T}m\Big|_{\mathcal{I}} = 0.$$

3. The global problem

This section is devoted to studying the global dynamics arising from Eguchi–Hanson-type initial data. The existence of a maximal development is guaranteed by theorem 1.7 and remark 1.8. In section 3.1, we specify our choice of coordinates on the orbits of the $SU(2) \times U(1)$ action and derive some geometric properties; here, we follow the exposition of [Daf04a] and [Daf04b] mutatis mutandis. Proving that the existence of a horizon would be contradictory is the content of sections 3.2 and 3.3.

3.1. Global biaxial Bianchi IX symmetry

Let $(\mathcal{S}, \bar{g}, K)$ be of Eguchi–Hanson type with negative mass M at infinity. Then, by theorem 1.7 and remark 1.8, there is a unique maximal forward development $(\mathcal{M}^+, g)$ which is asymptotically locally AdS. There is a projection map $\pi : \mathcal{M}^+ \to \mathcal{Q}^+$ onto a 2D manifold with boundary $\mathcal{Q}^+$ such that every $q \in \mathcal{Q}^+$ represents an orbit under the $SU(2) \times U(1)$ symmetry. The manifold $\mathcal{Q}^+$ can be embedded smoothly into $(\mathbb{R}^2, g_{\mathrm{Mink}})$ and its boundary consists of a

1D curve Σ (initial hypersurface) and a 1D curve Γ (central worldline, where $r = 0$). Choosing standard null coordinates (u, v) on $\mathbb{R}^{1+1}$, $\mathcal{Q}^+$ shall be endowed with a metric

$$h = -\frac{1}{2}\Omega^2(u, v)\,(\mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}u)\,.$$

We choose u such that the curves of constant u are outgoing and such that u as well as v are increasing to the future along Γ. A coordinate chart (u', v') preserves these assumptions if and only if

$$\frac{\partial u'}{\partial u} > 0,\ \frac{\partial v'}{\partial v} > 0,\ \frac{\partial u'}{\partial v} = \frac{\partial v'}{\partial u} = 0. \tag{3.1}$$

With respect to h, Σ is spacelike and Γ timelike. Conformal infinity $\mathcal{I} \subseteq \overline{\mathcal{Q}^+} \backslash \mathcal{Q}^+$ is defined as follows: Set

$$\mathcal{U} := \left\{ u \ :\ \sup_{(u,v)\in\mathcal{Q}^+} r(u, v) = \infty \right\}.$$

For each $u \in \mathcal{U}$, there is a unique $v^*(u)$ such that

$$(u, v^*(u)) \in \overline{\mathcal{Q}^+} \backslash \mathcal{Q}^+.$$

Note here that the closure is always taken with respect to the topology of $\mathbb{R}^2$. Now define null infinity as

$$\mathcal{I} := \bigcup_{u\in\mathcal{U}} (u, v^*(u)).$$

Since the spacetime is asymptotically locally AdS, null infinity $\mathcal{I}$ is timelike. We have

$$\mathcal{Q}^+ = D^+ (\Sigma \cup \mathcal{I})\,,$$

i.e. $\mathcal{Q}^+$ is in the future domain of dependence of Σ and $\mathcal{I}$. By a simple change of coordinates satisfying (3.1), we achieve that $u = v$ on $\mathcal{I}$. Note that in general, we cannot achieve that both $\mathcal{I}$ and Γ are straightened out in this way. We know that B extends continuously to $\mathcal{I}$ and vanishes there. Moreover, we know that the Hawking mass m extends continuously to $\mathcal{I}$ and equals a constant value $M < 0$.

Lemma 3.1. *The following hold:*

(i) r is unbounded on Σ.
(ii) $m \to M < 0$ as $r \to \infty$.
(iii) $r_v > 0$ for points in Σ with large r, and

$$\frac{r_v}{\Omega^2} \to c_0 > 0$$

as $r \to \infty$.

Proof. The radius r is unbounded on Σ by the definition of Eguchi–Hanson-type data. By $M < 0$, we immediately obtain that $m < 0$ on around $\mathcal{I}$. For the Hawking mass to be finite at infinity,

$$-4\frac{r_u r_v}{\Omega^2} = \frac{r^2}{\ell^2} + \mathcal{O}(r)$$

as $r \to \infty$. Moreover,

$$\frac{r_u}{r_v} \to -1$$

as $r \to \infty$ because we have chosen (u, v) such that $u = v$ on $\mathcal{I}$. The conformal factor Ω^2 grows as r^2 since the spacetime is asymptotically locally AdS. Therefore, we deduce that r_v/Ω^2 is positive and finite as $r \to \infty$. □

Proposition 3.2. *The above manifold $\mathcal{M}^+$ does not have any trapped or marginally trapped surfaces, i. e.*

$$r_u < 0 \text{ and } r_v > 0$$

globally in $\mathcal{Q}^+$.

Proof. Define the set

$$\mathcal{A} := \{(u, v) \in \mathcal{Q}^+ : m(u, v) < 0\}.$$

Evidently, $\mathcal{A}$ is open. From lemma 3.1, we know that the points of Σ with r sufficiently large are contained in $\mathcal{A}$. Call this set U_1. Moreover, since M is negative and m is continuous, $\mathcal{A}$ contains a neighbourhood of $\mathcal{I}$. Call this neighbourhood U_2. Let $\mathcal{C}$ be the connected component of $\mathcal{A}$ containing U_1 and U_2. Clearly, $\mathcal{C}$ is open as well. In $\mathcal{C}$, we have

$$0 > m = \frac{r^2}{2}\left(1 + 4\frac{r_u r_v}{\Omega^2}\right) + \frac{r^4}{2\ell^2}, \tag{3.2}$$

from which we conclude that $r_u r_v < 0$ wherever r is finite because $\mathcal{C}$ is connected. Since there is a point on Σ, contained in $\mathcal{C}$, where $r_v > 0$, we have that $r_u < 0$ and $r_v > 0$ in $\mathcal{C}$. From (A.7) and (A.8), we obtain $\partial_u m \leqslant 0$ and $\partial_v m \geqslant 0$ in $\mathcal{C}$, thus also in $\overline{\mathcal{C}}$. Therefore, $m < 0$ in $\overline{\mathcal{C}}$. Hence $\mathcal{C}$ is open and closed. We conclude that $\mathcal{Q}^+ = \mathcal{C} \subseteq \mathcal{A} \subseteq \mathcal{Q}^+$. The statement then follows from (3.2). □

Remark 3.3. The absence of anti-trapped surfaces can also be guaranteed by fixing the sign of r_u on Σ and $\mathcal{I}$ and then using (A.2).

This fact already allows us to prove a weak geometric statement about the potential singularities that can arise in the time evolution.

Definition 3.4. Let $p \in \overline{\mathcal{Q}^+}$. The indecomposable past subset $J^-(p) \cap \mathcal{Q}^+$ is said to be eventually compactly generated if there exists a compact subset $X \subseteq \mathcal{Q}^+$ such that

$$J^-(p) \subseteq D^+(X) \cup J^-(X).$$

Here we denote by $J^-(S)$ the causal past of a subset S.

Definition 3.5. A point $p \in \overline{\mathcal{Q}^+} \backslash \mathcal{Q}^+$ is a first singularity if $J^-(p) \cap \mathcal{Q}^+$ is eventually compactly generated and if any eventually compactly generated indecomposable past proper subset of $J^-(p) \cap \mathcal{Q}^+$ is of the form $J^-(q)$ for a $q \in \mathcal{Q}^+$.

Lemma 3.6. *Let $p \in \overline{\mathcal{Q}^+} \backslash \mathcal{Q}^+$ be a first singularity. Then*

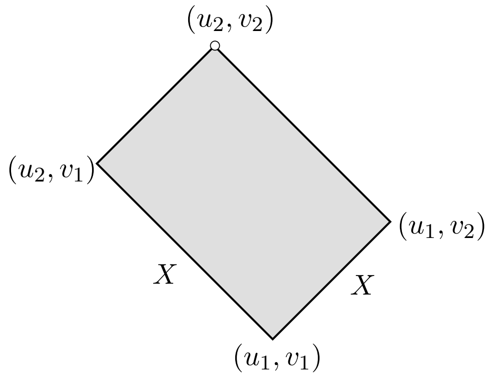

Figure 6. The extension property in the interior.

$$p \in \overline{\Gamma} \backslash \Gamma.$$

Proof. Suppose $p \notin \overline{\Gamma}$. Since the compact set X of definition 3.5 has to be wholly contained in $\mathcal{Q}^+$, we know that $p \notin \mathcal{I}$. In particular, p is the future endpoint of a rectangle, whose remainder is completely contained in the interior of $\mathcal{Q}^+$; see figure 6. By proposition 3.2, $r_u < 0$ and $r_v > 0$ in this rectangle. Therefore, we can apply the standard extension principle away from infinity and the central worldline—in a manner as e. g. in [Daf04a]—to conclude that $p \in \mathcal{Q}^+$, a contradiction. □

Theorem 3.7. *If $\mathcal{Q}^+ \backslash J^-(\mathcal{I}) \neq \emptyset$, then there is a null curve $\mathcal{H}^+ \subseteq \mathcal{Q}^+$ such that*

$$\mathcal{H}^+ = \overline{J^-(\mathcal{I})} \backslash \left(I^-(\mathcal{I}) \cup \overline{\mathcal{I}} \right). \tag{3.3}$$

The null curve $\mathcal{H}^+$ is the future event horizon.

Note that $I^-(S)$ denotes the chronological past of a subset S.

Proof. The horizon is given by

$$\mathcal{H}^+ = \mathring{J}^-(\mathcal{I}) \cap \mathcal{Q}^+.$$

Let p be the future endpoint of the horizon. Since $\mathcal{Q}^+ \backslash J^-(\mathcal{I}) \neq \emptyset$, $p \notin \overline{\Gamma} \backslash \Gamma$. Therefore p would not be the future endpoint of the horizon. If $p \in \Gamma$, there is an open neighbourhood U of p such that $U \cap \mathcal{Q}^+ \neq \emptyset$ If $p \notin \overline{\Gamma} \cup \overline{\mathcal{I}}$, then p is a first singularity and we have a contradiction to lemma 3.6. Therefore $p \in \overline{\mathcal{I}}$. □

Sections 3.2 and 3.3 are devoted to showing theorem 1.10.

3.2. An extension principle

We formulate and prove an extension principle tailored to extending a solution beyond a supposed horizon.

Theorem 3.8. *Let $(\tilde{r}, m, B)$ be a classical solution to the Einstein equation (2.2)–(2.5) in the punctured triangle $\Delta := \Delta_{d,u_0} \backslash \{(u_0 + d, u_0 + d)\}$. Let*

$$\mathcal{I}_{\text{aux}} = \overline{\Delta_{d,u_0}} \backslash \{\Delta_{d,u_0} \cup \{(u_0+d, u_0+d)\}\}.$$

Assume that

$$\tilde{r}\Big|_{\mathcal{I}_{\text{aux}}} = 0,\ m\Big|_{\mathcal{I}_{\text{aux}}} = M < 0,\ B\Big|_{\mathcal{I}_{\text{aux}}} = 0$$

and that

$$\lim_{v \to u_0+d} \tilde{r}(u_0+d, v) = 0.$$

Suppose that

$$\tilde{r}_u > 0 \text{ and } \tilde{r}_v < 0$$

in Δ, that

$$\inf_{\mathcal{I}_{\text{aux}}} \tilde{r}_u > 0 \text{ and } \sup_{\mathcal{I}_{\text{aux}}} \tilde{r}_v < 0$$

and that there is a $C > 0$ such that

$$\sup_{u_0 \leqslant v \leqslant u_0+d} \int_v^{u_0+d} \tilde{r}^{-3} \left(B^2 + (B_u)^2\right) \, \mathrm{d}u' + \sup_{\Delta} \left|\tilde{r}^{-1} \partial_u B\right| < C.$$

Then there is a $\delta > 0$ such that the solution $(\tilde{r}, m, B)$ can be extended to the strictly larger triangle $\Delta_{d+\delta,u_0}$.

Proof. Extending beyond the domain of existence means using the local well-posedness result to extend the solution further into the future. We will first need to make sure that on each constant v-slice, the function $\tilde{r}$ satisfies the correct boundary conditions. Reformulating equation (2.2), we obtain

$$\tilde{r}_{uv} = -\frac{4\tilde{r}_u \tilde{r}_v}{\tilde{r}^2 + 2|m|\tilde{r}^4 + \ell^{-2}} \tilde{r} \left(-1 + \frac{1}{3}R - 2m\tilde{r}^2\right).$$

Therefore $\tilde{r}_{uv} = 0$ on $\mathcal{I}_{\text{aux}}$. Using a coordinate change, we want to fix the value of $\tilde{r}_u$. By the assumptions and since $\tilde{r}_u = -\tilde{r}_v$ on $\mathcal{I}_{\text{aux}}$, $|\tilde{r}_u|, |\tilde{r}_v| \geqslant c > 0$. Thus

$$\frac{\mathrm{d}u'}{\mathrm{d}u}(u, v) = 2\tilde{r}_u(u, u),\ \frac{\mathrm{d}v'}{\mathrm{d}v} = 2\tilde{r}_v(v, v),$$

where $u' = u$ and $v' = v$ on $\mathcal{I}$, defines a regular change of coordinates that preserves the biaxial Bianchi IX symmetry. Moreover, in (u', v') coordinates,

$$\tilde{r}\Big|_{\mathcal{I}} = 0,\ \tilde{r}_{u'}\Big|_{\mathcal{I}} = \frac{1}{2},\ \tilde{r}_{u'v'}\Big|_{\mathcal{I}} = 0. \tag{3.4}$$

Hence we can assume without loss of generality that (3.4) already holds in the original (u, v) coordinates.

To increase the domain of existence, we also need initial v-slices of increased length. This can be achieved by an application of the standard local existence theorem away from infinity in double null coordinates whose proof proceeds by the same methods as for $\Lambda = 0$, which is standard by now. There is a $\delta_0 > 0$ such that we can extend the data to the slice $(u_0, u_0 + d + \delta']$ of constant $v = u_0$. Then, for all $\varepsilon > 0$, there is a $0 < \delta' < \delta_0$ such that there is a classical solution in the rectangle

$$\{u_0 + d \leqslant u \leqslant u_0 + d + \delta',\ u_0 \leqslant v \leqslant u_0 + d - \varepsilon\}\,.$$

Therefore, for all $\varepsilon > 0$, there is a $\delta' > 0$ such that the solution can be extended to the set

$$\Delta_\varepsilon := \Delta \cup (\Delta_{d+\delta',u_0} \cap \{v \leqslant u_0 + d - \varepsilon\})\,.$$

For each constant v-ray in Δ_ε, we have a initial data set, whose functions have norms uniformly bounded by $2C$. Note that the condition $1 - \mu > cr^2/\ell^2$ (for $c > 0$), required in [Dol17] holds everywhere because $M < 0$.

Therefore, by the local existence theorem of [Dol17] (theorem 3.2.5 there), there is a δ^* independent of ε such that each slice of constant v in Δ_ε yields a solution in a triangular domain of size δ^*. Now we choose $\varepsilon = \delta^*/2$ and see that the solution $(\tilde{r}, m, B)$ extends to a strictly larger triangle $\Delta_{d+\delta,u_0}$, where $\delta = \varepsilon$. □

The proof above yields another version of the extension principle that we formulate separately for the sake of clarity.

Corollary 3.9. *Suppose the assumptions of theorem 3.8 hold. Moreover, let us assume that the classical solution on Δ has an extension to the extended initial data slice $\tilde{\mathcal{N}} = (u_0, u_0 + d + \varepsilon]$. Then there is a $\delta > 0$ such that the solution $(\tilde{r}, m, B)$ can be extended to $\Delta_{d+\delta,u_0}$ such that it agree on $\tilde{\mathcal{N}} \cup \Delta_{d+u_0,u_0}$ with the given values. Furthermore, the extension is unique for sufficiently small $\delta > 0$*

3.3. A priori estimates

In this section, we first establish what was described through figure 4 in section 1.5 as the soft argument. This is the content of lemma 3.10. The remainder of the section contains the argument by contradiction, using the extension principle in form of corollary 3.9.

Lemma 3.10. *Let $\mathcal{Q}^+$ be as in theorem 1.10. Set $\Delta_u^d := \Delta_{d,u} \backslash \{(u + d, u + d)\}$ and $\mathcal{N}_u^d := \{v = u\} \cap \Delta_u^d$. Then for any $\Delta_{u_1}^{d_1} \subseteq \mathcal{Q}^+$, there is a $u_0 \geqslant u_1$ and $d_0 := d_1 - (u_0 - u_1)$ such that*

(i) $r \geqslant r_0 > 0$ in $\Delta_{u_0}^{d_0}$.
(ii) There are $q_1, q_2 > 0$ such that

$$q_1 \leqslant \frac{r_v}{\Omega^2} \leqslant q_2 \tag{3.5}$$

in $\Delta_{u_0}^d$. The constants q_1 and q_2 depend on the choice of r_0.

Proof. If $\Delta_{u_1}^{d_1}$ touches Γ, then by moving the initial slice of constant v to the future—as depicted in figure 4—, we achieve that $r \geqslant r_0 > 0$ since $r_v > 0$ globally by proposition 3.2. Let us choose r_0 maximal such that (i) holds.

By assumption, the bound on r_v/Ω^2 holds on Σ. Set

$$\mathcal{R} := \{r = r_0\} \cap \{u \leqslant u_0 + d_0\} \cap \mathcal{Q}^+.$$

The set $\mathcal{R}$ is closed and touches $\{u = u_0 + d + 0\}$ and Σ. The continuous function r_v/Ω^2 is positive in $\mathcal{Q}^+$ by proposition 3.2. Therefore the bound on r_v/Ω^2 holds in $\mathcal{R}$. We will show that (3.5) holds in the causal future of

$$S := \mathcal{R} \cup (\Sigma \cap \{r \geqslant r_0\})$$

such that the constants q_1 and q_2 depend on the values of r_v/Ω^2 on S. We rewrite the constraint equation (A.3) as

$$\partial_v \left(\frac{r_v}{\Omega^2}\right) = \left(-\frac{4r^3 r_u}{\Omega^2} (B_v)^2\right) \left(-\frac{2}{r^2(1-\mu)}\right) \frac{r_v}{\Omega^2}. \tag{3.6}$$

Given $(u, v) \in J^+(S)$, there is a $(u', v') \in S$ such that (u, v) and (u', v') are connected by a null curve. We integrate (3.6) along a ray of constant u to find

$$\begin{aligned}
\left(\frac{r_v}{\Omega^2}\right)(u, v) &= \exp\left(\int_{v'}^{v} \frac{4r^3 r_u}{\Omega^2} (B_v)^2 \frac{2}{r^2(1-\mu)} \mathrm{d}v''\right) \cdot \left(\frac{r_v}{\Omega^2}\right)(u', v') \\
&\geqslant \exp\left(-\frac{2}{r_0^2} \int_{u_0}^{v} \frac{-4r^3 r_u}{\Omega^2} (B_v)^2 \,\mathrm{d}v''\right) \cdot \left(\frac{r_v}{\Omega^2}\right)(u', v') \\
&\geqslant \exp\left(-\frac{2}{r_0^2} (m(u, v) - m(u', v'))\right) \cdot \left(\frac{r_v}{\Omega^2}\right)(u', v').
\end{aligned}$$

For the first inequality, we have used $1 - \mu > 1$ and $r \geqslant r_0$. For the second inequality, we have used (2.4) and have dropped a non-negative term. Therefore, we obtain

$$\mathrm{e}^{-\frac{2}{r_0^2}[M - m(u_0+d, u_0)]} \left.\frac{r_v}{\Omega^2}\right|_{(u',v')} \leqslant \left.\frac{r_v}{\Omega^2}\right|_{(u,v)} \leqslant \left.\frac{r_v}{\Omega^2}\right|_{(u',v')}. \tag{3.7}$$

This yields (3.5). □

Remark 3.11. A bound of the form (3.7) can always be achieved, independently of the exact value of M.

Now assume for the sake of contradiction that $\mathcal{Q}^+$ possesses a horizon $\mathcal{H}^+$. According to lemma 3.10, we find a $\Delta := \Delta_{u_0}^{d_0}$ such that $(u_0 + d, u_0) \in \mathcal{H}^+$ and such that the conclusions of the lemma hold. In particular, the constants and bounds will be fixed henceforth. Again, let $\mathcal{I}_{\mathrm{aux}} := \overline{\Delta_{d,u_0}} \backslash \{\Delta_{d,u_0} \cup \{(u_0 + d, u_0 + d)\}\}$. Let $\mathcal{N} := \mathcal{N}_{u_0}^{d_0}$. We always have that

$$\left.\tilde{r}\right|_{\mathcal{I}_{\mathrm{aux}}} = 0, \ \left.m\right|_{\mathcal{I}_{\mathrm{aux}}} = M < 0, \ \left.B\right|_{\mathcal{I}_{\mathrm{aux}}} = 0,$$

and that

$$r_u < 0 \text{ and } r_v > 0.$$

We will show that all the assumptions of the extension principle hold. Let us first turn to estimating the norms of B. The mass achieves its minimum at (u_1, u_0) with $u_1 = u_0 + d$ and its maximum on $\mathcal{I}_{\mathrm{aux}}$. Therefore, upon integration over constant v, we obtain

$$\int_v^{u_1} \left(-r r_u \left(1 - \frac{2}{3} R\right) + \frac{4}{\Omega^2} r^3 r_v (B_u)^2\right) \mathrm{d}u = M - m(u_1, v) \leqslant M - m(u_1, u_0).$$

Note that we have

$$1 - \frac{2}{3}R \geqslant \min\{B^2/2, 1\}.$$

We need to estimate the coefficients in the integral. From

$$\tilde{r}_u = \frac{1}{4r^2 \frac{r_v}{\Omega^2}}(1 - \mu)$$

and (3.6), we obtain

$$\frac{1}{4\ell^2}\left(\max_{u_0 \leqslant u \leqslant u_1} \frac{r_v}{\Omega^2}\bigg|_{(u,u_0)}\right)^{-1} \leqslant \tilde{r}_u \leqslant \mathrm{e}^{\frac{2}{r_0^2}[M - m(u_0+d,u_0)]}\left(\min_{u_0 \leqslant u \leqslant u_1} \frac{r_v}{\Omega^2}\bigg|_{(u,u_0)}\right)^{-1} \times \frac{\tilde{r}^2}{4}\left(1 + \frac{2|M|}{r^2} + \frac{r^2}{\ell^2}\right) \tag{3.8}$$

and see that $\tilde{r}_u$ is uniformly bounded above and below by a constant depending only on data on $\mathcal{N}$. Therefore

$$C_1(u - v) \leqslant \tilde{r} \leqslant C_2(u - v)$$

and

$$\lim_{v \to u_0 + d} \tilde{r}(u_0 + d, v) = 0. \tag{3.9}$$

Furthermore,

$$-r_u\bigg|_{(u,v)} = \frac{1 + \frac{2|m|}{r^2} + \frac{r^2}{\ell^2}}{4\frac{r_v}{\Omega^2}} \geqslant \frac{r^2}{4\ell^2} \frac{1}{\max_{u_0 \leqslant u \leqslant u_1} \frac{r_v}{\Omega^2}\bigg|_{(u,u_0)}}.$$

Therefore there is a constant C_u depending only on values of $\tilde{r}$, $\tilde{r}_v$ and m on $v = u_0$ such that

$$\int_v^{u_1} r^3 \left(\min\{B^2, 1\} + (B_u)^2\right) \, \mathrm{d}u < C_u \tag{3.10}$$

uniformly in Δ.

Thus,

$$|B(u,v)| \leqslant \left(\int_v^u \frac{1}{r^3} \, \mathrm{d}u'\right)^{1/2} \left(\int_v^u r^3 B_u^2 \, \mathrm{d}u'\right)^{1/2} \leqslant \frac{C_2^{1/2}}{2} C_u^{1/2} (u - v)^2.$$

It follows that

$$\tilde{r}^{-2}|B| \leqslant C_{\text{pointwise}} \tag{3.11}$$

uniformly. Together with (3.10), this yields

$$\int_u^{u_1} r^3 \left(B^2 + (B_u)^2\right) \, \mathrm{d}u < C'_u. \tag{3.12}$$

In a similar way, one also obtains

$$\int_{u_0}^{v} r^3 \left(B^2 + (B_v)^2\right) \, \mathrm{d}v < C'_v$$

for $v \in [u_0, u_1)$ from integrating $\partial_v m$ and then deriving a bound on r_u/Ω^2 as (3.7). Here one uses that

$$\frac{r_u}{\Omega^2}\bigg|_{\mathcal{I}} = -\frac{r_v}{\Omega^2}\bigg|_{\mathcal{I}}.$$

Using the wave equation for B in the form

$$\partial_v\left(r^{3/2}B_u\right) = -\frac{3}{2}r^{1/2}r_uB_v - \frac{\Omega^2}{3r^{1/2}}\left(\mathrm{e}^{-2B} - \mathrm{e}^{-8B}\right) \tag{3.13}$$

yields

$$\begin{aligned}\left|r(u,v)^{3/2}B_u(u,v)\right| \leqslant & r(u,u_0)^{3/2}|B_u(u,u_0)| + C\int_{u_0}^{v} r(u,v')^{5/2}|B_v(u,v')|\,\mathrm{d}v' \\ &+ \frac{1}{3}\left(\int_{u_0}^{v}\frac{\Omega^2}{r^2}\,\mathrm{d}v'\right)^{1/2}\left(\int_{u_0}^{v}\Omega^2 r\left|\mathrm{e}^{-2B} - \mathrm{e}^{-8B}\right|^2\,\mathrm{d}v'\right)^{1/2}.\end{aligned}$$

Since Ω^2/r^2 is bounded by virtue of the bounds established above, the third term is easily seen to be bounded. The second term is estimates as

$$\begin{aligned}\int_{u_0}^{v} r^{5/2}|B_v|\,\mathrm{d}v' \leqslant & \left(\int_{u_0}^{v} r^3(B_v)^2\,\mathrm{d}v'\right)^{1/2}\left(\int_{u_0}^{v} r^2\mathrm{d}v'\right)^{1/2} \\ \leqslant & C\left(\int_{u_0}^{v}\frac{\partial_v r}{(-\tilde{r}_v)}\,\mathrm{d}v'\right)^{1/2} \\ \leqslant & Cr(u,v)^{1/2}.\end{aligned}$$

Therefore,

$$|rB_u| \leqslant r(u,u_0)|B_u(u,u_0)| + C,$$

where C depends on values on $\mathcal{N}$ and on C_1, C_2,, C'_v. Since B is a classical solution up to and including the horizon, there is an $\alpha > 0$ such that

$$B = s^{\alpha}\left(a_0(t) + a_1(t)s + o(s)\right) \tag{3.14}$$

for smoothly differentiable functions a_0 and a_1 of $t = (u+v)/2$. Here $s = (u-v)/2$. From above, the asymptotics of r, r_u, r_v and Ω^2 are known as we approach the boundary. Inserting (3.14) into (3.13), we obtain $\alpha = 4$. Therefore, rB_u is bounded on $\mathcal{N}$ and we have established the desired pointwise bound on rB_u in Δ.

An application of the extension principle (theorem 3.8) yields theorem 1.10 if it also holds true that

$$\inf_{\mathcal{I}_{\text{aux}}}\tilde{r}_u > 0 \text{ and } \sup_{\mathcal{I}_{\text{aux}}}\tilde{r}_u < 0.$$

This has been established already in (3.8), thus finishing the proof of theorem 1.10.

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Appendix. Biaxial Bianchi IX symmetry

A.1. Formulae related to the renormalised Hawking mass

In this section, we collect some useful calculations and identities related to the renormalised Hawking mass. Let us first start generally with an n-dimensional Lorentzian manifold $(\mathcal{M}, g)$ with Levi-Civita connection ∇ and a spacelike hypersurface $(\mathcal{N}, \overline{g})$ with induced second fundamental form K. Let n be the timelike normal on $\mathcal{N}$. Let (Σ, γ) be a compact 3D submanifold of $\mathcal{N}$, separating $\mathcal{N}$ into an inside and an outside. Let ν be the unit normal pointing outside. Set $l_\pm := n \pm \nu$. For X and Y tangent to Σ, we define the symmetric null second forms

$$\chi_\pm(X, Y) := g\left(\nabla_X l_\pm, Y\right).$$

Clearly,

$$\chi_\pm = \frac{1}{2}\mathcal{L}_{l_\pm} g.$$

The associated null expansion scalars are defined by

$$\vartheta_\pm := \mathrm{tr}_\gamma \chi_\pm = \gamma^{AB}\left(\chi_\pm\right)_{AB} = \gamma^{AB}\nabla_A\left(l_\pm\right)_B.$$

Let H be the mean curvature of Σ. Then we immediately obtain

$$\vartheta_\pm = \mathrm{tr}_\gamma K \pm H.$$

Henceforth, we will consider only dimension $n = 5$. Let us assume that $\mathcal{N}$ is foliated by topological 3-spheres Σ_r, where

$$4\pi^2 r = \mathrm{Vol}\left(\Sigma_r\right) = \int_{\Sigma_r} \mathrm{d}\mu_\gamma = \int_{S^3} \sqrt{\gamma}\, \mathrm{d}\mu_{S^3}.$$

We compute

$$\mathcal{L}_{l_\pm}\left(\mathrm{Vol}\left(\Sigma_r\right)\right) = \int_{S^3} \frac{1}{2\sqrt{\gamma}}\mathcal{L}_{l_\pm}\left(\det\gamma\right)\, \mathrm{d}\mu_{S^3} = \int_{S^3} \frac{1}{2}\gamma^{AB}\left(\mathcal{L}_{l_\pm}\gamma\right)_{AB}\sqrt{\gamma}\, \mathrm{d}\mu_{S^3} = \int_{\Sigma_r} \vartheta_\pm\, \mathrm{d}\mu_{\Sigma_r}.$$

Assuming that $\vartheta_\pm$ is constant on Σ_r, we obtain

$$l_\pm r = \frac{r}{3}\mathrm{tr}_\gamma \chi_\pm.$$

Therefore,

$$g(\nabla r, \nabla r) = -\frac{r^2}{12}\left(\mathrm{tr}_\gamma \chi_+\right)\left(\mathrm{tr}_\gamma \chi_-\right) = -\frac{r^2}{12}\left(\left(\mathrm{tr}_\gamma K\right)^2 - H^2\right). \tag{A.1}$$

A.2. The Einstein vacuum equations reduced by biaxial Bianchi IX symmetry

We quote the following result from [DH06a] and do not present a derivation:

Theorem A.1. *Let $(\mathcal{M}, g)$ exhibit a biaxial Bianchi IX symmetry with metric*

$$g = -\frac{1}{2}\Omega^2\left(\mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}u\right) + \frac{1}{4}r^2 \mathrm{e}^{2B}\left(\sigma_1^2 + \sigma_2^2\right) + \frac{1}{4}r^2 \mathrm{e}^{-4B}\,\sigma_3^2$$

Then the Einstein vacuum equation (1.2) for $\Lambda = -6/\ell^2 < 0$, understood as a classical system of partial differential equations, are equivalent to the system of two constraint equations

$$\partial_u \left(\frac{r_u}{\Omega^2} \right) = -\frac{2r}{\Omega^2} \left(B_u \right)^2 \tag{A.2}$$

$$\partial_v \left(\frac{r_v}{\Omega^2} \right) = -\frac{2r}{\Omega^2} \left(B_v \right)^2 \tag{A.3}$$

and four evolution equations

$$r_{uv} = -\frac{\Omega^2 R}{3r} - \frac{2r_u r_v}{r} - \frac{\Omega^2 r}{\ell^2} \tag{A.4}$$

$$(\log \Omega)_{uv} = \frac{\Omega^2 R}{2r^2} + \frac{3}{r^2} r_u r_v - 3B_u B_v + \frac{\Omega^2}{2\ell^2} \tag{A.5}$$

$$B_{uv} = -\frac{3}{2}\frac{r_u}{r} B_v - \frac{3}{2}\frac{r_v}{r} B_u - \frac{\Omega^2}{3r^2} \left(\mathrm{e}^{-2B} - \mathrm{e}^{-8B} \right), \tag{A.6}$$

where

$$R = 2\mathrm{e}^{-2B} - \frac{1}{2}\mathrm{e}^{-8B}$$

is the scalar curvature of the group orbits.

Proof. One needs to compute the components of the Ricci curvature. The constraints are the uu and vv components. The equation (A.5) comes from the uv component. The equations (A.4) and (A.6) are the content of the remaining components. □

Now our aim is to reformulate the equations of theorem A.1 in terms of the renormalised Hawking mass. The derivatives of the mass are given by

$$\partial_u m = r r_u \left(1 - \frac{2}{3} R \right) - \frac{4}{\Omega^2} r^3 r_v (B_u)^2 \tag{A.7}$$

$$\partial_v m = r r_v \left(1 - \frac{2}{3} R \right) - \frac{4}{\Omega^2} r^3 r_u (B_v)^2. \tag{A.8}$$

Since $x \mapsto 2\mathrm{e}^{-2x} - \mathrm{e}^{-8x}/2$ is positive for $|x| \to \infty$ and has no extremum, we conclude that $\partial_u m \leqslant 0$ and $\partial_v m \geqslant 0$.

Proposition A.2. *Assume that (A.4), (A.6)–(A.8) hold. Moreover, define Ω via (2.1). Then the constraints (A.2) and (A.3) hold. If the right hand side of (A.4) can be differentiated in u, then also (A.5) holds.*

Proof. The proof is a calculation. We obtain

$$\frac{4r_u}{\Omega^2} = -\frac{1}{r_v} + \frac{2m}{r^2 r_v} - \frac{r^2}{\ell^2 r_v}$$

from (2.1). This yields

$$\partial_u\left(\frac{r_u}{\Omega^2}\right)=-\frac{r_u}{r_v}\frac{r_{uv}}{\Omega^2}+\frac{1}{2}\frac{m_u}{r^2r_v}-\frac{mr_u}{r^3r_v}-\frac{1}{2}\frac{r}{\ell^2}\frac{r_u}{r_v}$$
$$=\frac{r_u}{r_v}\left(\frac{1}{2r}-\frac{R}{3r}+\frac{R}{3r}-\frac{1}{2r}+\frac{m}{r^3}-\frac{r}{2\ell^2}+\frac{r}{\ell^2}-\frac{1}{2}\frac{r}{\ell^2}-\frac{m}{r^3}\right)-\frac{2}{\Omega^2}r\left(B_u\right)^2$$
$$=-\frac{2}{\Omega^2}r\left(B_u\right)^2.$$

The second constraint equation is obtained analogously. To obtain the equation for $(\log\Omega)_{uv}$, we multiply (A.2) by Ω^2 and differentiate with respect to u:

$$-2r_u\left(\log\Omega\right)_{uv}+r_{uuv}-2r_{uv}\frac{\Omega_u}{\Omega}=-2r_v\left(B_u\right)^2-4rB_uB_{uv}.$$

Using (A.4), we obtain

$$r_{uuv}-2r_{uv}\frac{\Omega_u}{\Omega}=\frac{\Omega^2}{r^2}r_uR+\frac{4\Omega^2}{3r}B_u\left(\mathrm{e}^{-2B}-\mathrm{e}^{-8B}\right)+\frac{\Omega^2r_u}{\ell^2}+6\frac{(r_u)^2}{r^2}r_v+4r_v(B_u)^2.$$

Applying (A.6), the desired equation follows. □

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